

Affine cellularity of affine Brauer algebras

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Abstract

We show that the affine Brauer algebras are affine cellular algebras in the sense of Koenig and Xi.

Keywords: Brauer algebras; Colored Brauer diagrams; Affine Brauer algebras; Affine cellular algebras

1 Introduction

In order to approach the fundamental problem of classifying the irreducible representations of a given finite-dimensional algebra, the concept of cellularity, defined by Graham and Lehrer in [GL], has proven extremely useful. Examples of cellular algebras include many finite-dimensional Hecke algebras.

Recently, Koenig and Xi in [KX3] has generalized this concept to algebras over a noetherian domain k of not necessarily finite dimension, by introducing the notion of an affine cellular algebra. The most important class of examples of affine cellular algebras, which has been discussed in [KX2], is given by the extended affine Hecke algebras of type A . Recently, Guilhot and Miemietz have proved that affine Hecke algebras of rank two with generic parameters are affine cellular in [GuM]; Kleshchev and Loubert have proved that KLR algebras of finite type are affine cellular in [KL]. In [C1] and [C2], we have shown that the BLN-algebras which have been introduced by McGerty in [Mc], and the affine q -Schur algebras which have been defined by Lusztig in [L], are affine cellular algebras.

The Brauer algebra is introduced as an enlargement of the symmetric group algebra and is in Schur-Weyl duality with the orthogonal or symplectic group. The BMW algebra, which is introduced in [BW] and [Mu], can be considered as a deformation of the Brauer algebra by replacing the symmetric groups by their Hecke algebras. In [MW], they have constructed a basis of the BMW algebra, which is indexed by Brauer n -diagrams. In [Xi], using this basis, he has shown that the basis is in fact a cellular basis and thus the BMW algebra is cellular (see also [E]).

The cyclotomic Brauer algebra is a corresponding enlargement of the complex reflection group algebra of type $G(m, 1, n)$. This has been introduced by Häring-Oldenburg in [HO] as a specialization of the cyclotomic BMW algebra, and has been studied by various authors (see for example [RY, RX, BCV]). In [RY], they have given a cellular basis of the cyclotomic Brauer algebra and thus it is cellular (see also [BCV]).

The affine Brauer algebra has been introduced by Häring-Oldenburg in [HO] as a specialization of the affine BMW algebra, and has been studied in [GH]. In [GH], they have introduced the colored Brauer n -diagrams, using which they have defined the affine Brauer algebra. In the subsequent papers [GM] and [G], they have constructed several bases of the affine BMW algebras which are all indexed by the colored Brauer n -diagrams.

In this note, We will show that the affine Brauer algebras are affine cellular algebras in the sense of Koenig and Xi. Our proof relies on the fact that the group algebra of the extended affine Weyl group of type A is an affine cellular algebra over \mathbb{Z} .

The organization of this note is as follows. In Section 2, we introduce affine cellular algebras. In Section 3, we will recall the definitions of Brauer algebras and affine Brauer algebras. In Section 4, we prove our main result Theorem 4.1.

2 Affine cellular algebras

Let k be a noetherian domain. For a k -algebra A , a k -linear anti-automorphism i of A satisfying $i^2 = id_A$ will be called a k -involution on A . For two k -modules V and W , we denote by τ the map $V \otimes W \rightarrow W \otimes V$ given by $\tau(v \otimes w) = w \otimes v$. If $B = k[x_1, \dots, x_t]/I$ for some ideal I in a polynomial ring in finitely many variables x_1, \dots, x_t over k , then B is called an affine k -algebra.

Definition 2.1. (see [KX3, Definition 2.1]) Let A be a unitary k -algebra with a k -involution i . A two-sided ideal J in A is called an affine cell ideal if and only if the following data are given and the following conditions are satisfied:

- (1) We have $i(J) = J$.
- (2) There exist a free k -module of finite rank and an affine k -algebra B with a k -involution σ such that $\Delta := V \otimes_k B$ is an A - B -bimodule, where the right B -module structure is induced by the right regular B -module B_B .
- (3) There is an A - A -bimodule isomorphism $\alpha : J \rightarrow \Delta \otimes_B \Delta'$, where $\Delta' = B \otimes_k V$ is a B - A -bimodule with the left B -module induced by the left regular B -module ${}_B B$ and with the right A -module structure defined by $(b \otimes v)a := \tau(i(a)(v \otimes b))$ for $a \in A$, $b \in B$ and $v \in V$, such that the

following diagram is commutative:

$$\begin{array}{ccc}
j & \xrightarrow{\alpha} & \Delta \otimes_B \Delta' \\
i \downarrow & & \downarrow v \otimes b \otimes_B b' \otimes w \mapsto w \otimes \sigma(b') \otimes_B \sigma(b) \otimes v \\
J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta'
\end{array}$$

The algebra A together with its k -involution i is called affine cellular if and only if there is a k -module decomposition $A = J'_1 \oplus J'_2 \oplus \cdots J'_n$ (for some n) with $i(J'_l) = J'_l$ for $1 \leq l \leq n$, such that, setting $J_m := \bigoplus_{l=1}^m J'_l$, we have a chain of two-sided ideals of A : $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$, where each $J'_m = J_m/J_{m-1}$ ($1 \leq m \leq n$) is an affine cell ideal of A/J_{m-1} (with respect to the involution induced by i on the quotient).

To prove that an algebra is affine cellular, one canonical way is to use the definitions. The following lemma, which can be regarded as an affine version of the iterated inflation developed in [KX1] and [KX2], provides another possibility to verify the affine cellularity of a given algebra, and describes also the structures of a general affine cellular algebra.

Lemma 2.1. *Let k be a noetherian domain, A a unitary k -algebra with a k -involution i . Suppose that there is a decomposition*

$$A = \bigoplus_{j=1}^m V_j \otimes_k V_j \otimes_k B_j \quad (\text{direct sums of } k\text{-modules})$$

where V_j is a free k -module of finite rank and B_j is an affine cellular algebra with respect to an involution σ_j and a cell chain $J_1^{(j)} \subset J_2^{(j)} \subset \cdots J_{s_j}^{(j)} = B_j$ for each j . Define $J_t = \bigoplus_{j=1}^t V_j \otimes_k V_j \otimes_k B_j$. Assume that the restriction of i on $V_j \otimes_k V_j \otimes_k B_j$ is given by $w \otimes v \otimes b \mapsto v \otimes w \otimes \sigma_j(b)$. If for each j there is a bilinear form $\phi_j : V_j \otimes_k V_j \rightarrow B_j$ such that $\sigma_j(\phi_j(w, v)) = \phi_j(v, w)$ for all $w, v \in V_j$ and that the multiplication of two elements in $V_j \otimes_k V_j \otimes_k B_j$ is governed by ϕ_j modulo J_{j-1} , that is, for $x, y, u, v \in V_j$ and $b, c \in B_j$, we have $(x \otimes y \otimes b)(u \otimes v \otimes c) = x \otimes v \otimes b\phi_j(y, u)c$ modulo the ideal J_{j-1} , and if $V_j \otimes V_j \otimes J_l^{(j)} + J_{j-1}$ is an ideal in A for all l and j , then A is an affine cellular algebra.

Proof. Since $J_1^{(j)} \subset J_2^{(j)} \subset \cdots J_{s_j}^{(j)} = B_j$, $j = 1, 2, \dots, m$ is a cell chain for the given affine cellular algebra B_j , we can check that the following chain of ideals in A satisfies all conditions in Definition 2.1:

$$\begin{aligned}
& V_1 \otimes V_1 \otimes J_1^{(1)} \subset \cdots \subset V_1 \otimes V_1 \otimes J_{s_1}^{(1)} \subset V_1 \otimes V_1 \otimes B_1 \oplus V_2 \otimes V_2 \otimes J_1^{(2)} \\
& \subset V_1 \otimes V_1 \otimes B_1 \oplus V_2 \otimes V_2 \otimes J_2^{(2)} \subset \cdots \subset V_1 \otimes V_1 \otimes B_1 \oplus V_2 \otimes V_2 \otimes B_2 \\
& \subset \cdots \subset \bigoplus_{j=1}^{m-1} V_j \otimes V_j \otimes B_j \oplus V_m \otimes V_m \otimes J_1^{(m)} \subset \cdots \\
& \subset \bigoplus_{j=1}^{m-1} V_j \otimes V_j \otimes B_j \oplus V_m \otimes V_m \otimes J_{s_m}^{(m)} = A.
\end{aligned}$$

In fact, we take a fixed non-zero element $v_j \in V_j$ and suppose that $\alpha : J_t^{(j)} \rightarrow (V_t^{(j)} \otimes B_t^{(j)}) \otimes_{B_t^{(j)}} (B_t^{(j)} \otimes V_t^{(j)})$ is the B_j -bimodule isomorphism in the definition of the affine cell ideal $J_t^{(j)}$ for the affine cellular algebra B_j . Define

$$\begin{aligned} \beta : V_j \otimes V_j \otimes J_t^{(j)} &\rightarrow (V_j \otimes v_j \otimes V_t^{(j)} \otimes B_t^{(j)}) \otimes_{B_t^{(j)}} (B_t^{(j)} \otimes V_j \otimes v_j \otimes V_t^{(j)}) \\ u \otimes v \otimes x &\rightarrow \sum_l (u \otimes v_j \otimes x_l^{(1)} \otimes b_l^{(1)}) \otimes (b_l^{(2)} \otimes v \otimes v_j \otimes x_l^{(2)}), \end{aligned}$$

where $u, v \in V_j$, $x \in J_t^{(j)}$ and $\alpha(x) = \sum_l (x_l^{(1)} \otimes b_l^{(1)}) \otimes (b_l^{(2)} \otimes x_l^{(2)})$. Then one can verify that β is an A -bimodule isomorphism and makes the corresponding diagram in the definition of affine cell ideals commutative. Here $V_j \otimes V_j \otimes J_t^{(j)}$ is an affine cell ideal in the corresponding quotient of A . Thus A is an affine cellular algebra. \square

3 Brauer algebras and Affine Brauer algebras

3.1 Brauer algebras

In this subsection, we recall the definition of Brauer algebras and introduce some notations for the later use.

Let $n \in \mathbb{N}$, and let $\mathbb{Z}[\delta]$ be the polynomial ring in one variable δ over the integers.

The Brauer algebra $B_{\mathbb{Z}[\delta]}(n, \delta)$, or written briefly $B(n, \delta)$, has as $\mathbb{Z}[\delta]$ -linear basis the set of all partitions of the set $S = \{1, 2, \dots, n, 1', 2', \dots, n'\}$ into two-element subsets (here the cardinality of S is $2n$). As usual we shall represent the basis element by a diagram in a rectangle of the plane, where the top row has n vertices marked by $1, 2, \dots, n$ from left to right; and the bottom row is numbered by $1', 2', \dots, n'$ from left to right. If i and j are in the same subset, we draw a line between i and j . We call the corresponding diagram a Brauer n -diagram and denote by B_n the set of all Brauer n -diagrams.

The multiplication in the Brauer algebra $B(n, \delta)$ is just the concatenation of two diagrams with a coefficient counting the number of cycles produced by forming the concatenation. Let D_1 and D_2 be two Brauer n -diagrams, we can compose D_1 and D_2 to get another Brauer n -diagram $D_1 \circ D_2$ by replacing D_1 and D_2 and joining the corresponding points; interior loops are deleted. Let r denote the number of deleted loops, then we define $D_1 \cdot D_2 = \delta^r D_1 \circ D_2$.

We define an order on S by $1 < 2 < \dots < n, n' < (n-1)' < \dots < 2' < 1'$, and $i < j'$ for all $1 \leq i, j \leq n$. For $d \in B_n$, we write $t(d)$ for the number of ‘through strings’, that is, vertical lines which connect a top vertex with a bottom vertex.

Generally, given an arbitrary ring R with identity and an element $\delta \in R$, we can define the Brauer algebra $B_R(n, \delta)$ over R by using the Brauer n -diagrams as R -basis. The following result is well-known in [GL]; see also [KX2].

Theorem 3.1. *The Brauer algebra $B_R(n, \delta)$ is cellular for any ring R and $\delta \in R$.*

3.2 Affine Brauer algebras

In this subsection, we will recall the definition of the affine Brauer algebras given in [GH], which can be considered as a sort of ‘wreath product’ of \mathbb{Z} with the Brauer algebras.

We define a colored Brauer n -diagram, which is a Brauer diagram in which each strand is endowed with an orientation and labeled by an element of the group \mathbb{Z} . We denote by \widehat{B}_n the set of all colored Brauer n -diagrams. Two labelings are regarded as the same if the orientation of a strand is reversed and the group element associated to the strand is inverted. Note that we only consider these diagrams without closed loops inside the frame.

Given $n \in \mathbb{N}$, let $R = \mathbb{Z}[\delta_0, \delta_1, \dots]$ be the ring with infinite variables $\delta_0, \delta_1, \dots$ over the integers. We now define the affine Brauer algebra $\widehat{B}_{n,R}$ over R . As a vector space, $\widehat{B}_{n,R}$ is the R -span of all colored Brauer n -diagrams. We define the product $x \cdot y$ of two colored Brauer n -diagrams x and y using the concatenation of x above y , where we identify the bottom nodes of x and the top nodes of y . More precisely, we first choose compatible orientations of the strands of x and y . Then we concatenate the diagrams and add the labels on each strand of the new diagram. Any closed loop in this new diagram can be oriented such that as the strand passes through the leftmost central node in the loop it points downwards. If this oriented loop is labeled by $\pm i$ for some $i \in \mathbb{N}$, then the diagram is set to be equal to δ_i times the same diagram with the loop moved.

Let z be the colored Brauer n -diagram obtained by removing all the closed loops, and let m_i be the number of deleted loops with labels $\pm i$ for each $i \in \{0, 1, 2, \dots\}$, then we define $x \cdot y = (\delta_0^{m_0} \delta_1^{m_1} \delta_2^{m_2} \dots)z$.

The affine Brauer algebra $\widehat{B}_{n,R}$ has an explicit presentation as follows. It can also be defined as the R -algebra generated by $\{s_i, e_i, t_j \mid 1 \leq i \leq n-1, \text{ and } 1 \leq j \leq n\}$ subject to the following relations:

- (a) $s_i^2 = 1$, for $1 \leq i \leq n-1$.
- (b) $s_i s_j = s_j s_i$, if $|i - j| > 1$.
- (c) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $1 \leq i < n-1$.
- (d) $s_i t_j = t_j s_i$, if $j \neq i, i+1$.
- (e) $s_i e_j = e_j s_i$, if $|i - j| > 1$.
- (f) $e_i e_j = e_j e_i$, if $|i - j| > 1$.
- (g) $e_i t_j = t_j e_i$, if $j \neq i, i+1$.
- (h) $t_i t_j = t_j t_i$, for $1 \leq i, j \leq n$.
- (i) $s_i t_i = t_{i+1} s_i$, for $1 \leq i < n$.

- (j) $e_i s_i = e_i = s_i e_i$, for $1 \leq i \leq n-1$.
- (k) $s_i e_{i+1} e_i = s_{i+1} e_i$, for $1 \leq i \leq n-2$.
- (l) $e_{i+1} e_i s_{i+1} = e_{i+1} s_i$, for $1 \leq i \leq n-2$. (m) $e_i e_j e_i = e_i$, if $|i-j|=1$.
- (n) $e_i t_i t_{i+1} = e_i = t_i t_{i+1} e_i$, for $1 \leq i \leq n-1$.
- (o) $e_i t_i^a e_i = \delta_a e_i$, for $a \geq 0$ and $1 \leq i \leq n-1$.

One can prove that the two definitions of $\widehat{B}_{n,R}$ are equivalent by the arguments similar to those for BMW algebras in [MW].

4 Affine cellularity of affine Brauer algebras

In this section, we will show that $\widehat{B}_{n,R}$ is an affine cellular algebra.

Definition 4.1. Suppose that $n, k \in \mathbb{N}$. An flat (n, k) -dangle is a partition of $\{1, 2, \dots, n\}$ into k one-element subsets, and $(n-k)/2$ two-element subsets, here k must be a natural number in $\{n, n-2, n-4, \dots\}$. An (n, k) -dangle is a flat (n, k) -dangle to which an integer $r \in \mathbb{Z}$ has been assigned to every subset of size 2.

We can represent a flat (n, k) -dangle d by a set of n nodes labeled by the set $\{1, 2, \dots, n\}$, where there is an arc joining i to j if $\{i, j\} \in d$, and there is a vertical line starting from i if $\{i\} \in d$. An (n, k) -dangle can be represented graphically by first labeling each arc of the underlying (n, k) -dangle and then giving it the following orientation: we let all one-element sets have a downward orientation and all two-element sets have a right orientation.

We denote by $D(n, k)$ the set of all (n, k) -dangles, and let $V(n, k)$ denote the vector space spanned by all (n, k) -dangles. We also define $V'(n, k)$ be a copy of $V(n, k)$, but draw the pictures dangling upward rather than down, and label the vertices by $1', 2', \dots, n'$. For each element $d \in D(n, k) \subset V(n, k)$, let $d' \in V'(n, k)$ be the corresponding element. For each $k \in \mathbb{N}$, we denote by A_k the group algebra of the wreath product of \mathbb{Z} and the symmetric group Σ_k .

Lemma 4.1. *For each element $d \in \widehat{B}_n$, we can write d uniquely as an element in $V(n, k) \otimes_R V'(n, k) \otimes_R A_k$ for some $k \in \mathbb{N}$.*

Proof. Suppose that there are k vertical lines in $d \in \widehat{B}_n$. Then we have two (n, k) -dangles d_1, d'_2 (by cutting off all vertical lines) and an element $\tilde{d} \in A_k$. These data (d_1, d'_2, \tilde{d}) are uniquely determined by d , where \tilde{d} is obtained in the following way: we can easily get a permutation $\pi(d) \in \Sigma_k$, if we also consider the labels on the k vertical lines then we get an element of A_k , which is just \tilde{d} . Thus we can write d as $d_1 \otimes d'_2 \otimes \tilde{d}$. Clearly, if we are given $d_1, d'_2 \in D(n, k)$, and $\tilde{d} \in A_k$, we have a unique element $d \in \widehat{B}_n$ with the data (d_1, d'_2, \tilde{d}) . \square

Now we want to define an R -linear form φ_k from $V'(n, k) \otimes_R V(n, k)$ to A_k . Given two basis elements $d'_1 \in V'(n, k), d_2 \in V(n, k)$. Then by Lemma 4.1, we can form the element $X_j := d_j \otimes d'_j \otimes 1$ in $V(n, k) \otimes_R V'(n, k) \otimes_R A_k$ for $j = 1, 2$. Suppose that the product $X_1 \cdot X_2$ is expressed as an R -linear combination of the basis elements in $\{d \mid d \in \widehat{B_n}\}$, say $\sum_j f'_j c_j$, where $f'_j \in R$. It is clear from the multiplication in $\widehat{B_{n,R}}$ that this expression of $X_1 \cdot X_2$ can be written as $\sum f_i c_i + a$ with $t(c_i) = k$ and a in the R -spanning of those basis elements c with $t(c) < k$. Now we rewrite c_i as the form $c_{i,1} \otimes c'_{i,2} \otimes \tilde{c}_i$. Moreover, for those c_i with $t(c_i) = k$, we know from the concatenation of two tangles that they are of the form $d_1 \otimes d'_2 \otimes \tilde{c}_i$. Here $X_1 \cdot X_2$ can be written as $\sum d_1 \otimes d'_2 \otimes f_i \tilde{c}_i + a$. Now we can define $\varphi_k(d'_1, d_2) := \sum_i f_i \tilde{c}_i \in A_k$.

Now we define J_k to be the R -module generated by basis elements d with $t(d) \leq k$. It is clear that $J_k \subset J_{k+1}$ and J_k is an ideal in $\widehat{B_{n,R}}$ for all k .

Lemma 4.2. *If $c = c_1 \otimes c'_2 \otimes \tilde{c}$ and $d = d_1 \otimes d'_2 \otimes \tilde{d}$ with $c_1, d_1 \in V(n, k), c'_2, d'_2 \in V'(n, k)$ and $\tilde{c}, \tilde{d} \in A_k$, then $c \cdot d = c_1 \otimes d'_2 \otimes \tilde{c} \varphi_k(c'_2, d_1) \tilde{d} \pmod{J_{k-2}}$.*

Proof. We may consider c'_2 as a (k, n) -dangle, and we now consider the concatenation of the two tangles c'_2 and d_1 , this gives us a (k, k) -dangle T . If we write this tangle T as a linear combination of the basis elements $\{x \mid x \in \widehat{B_k}\}$, then we see that $T = I_k \otimes I_k \otimes \varphi_k(c'_2, d_1) + a'$, where $\varphi_k(c'_2, d_1)$ is the bilinear form defined as above, $a' \in J_{k-2}$ and $I_k = id_k$ is the (k, k) -dangle with k vertical strands. So the product $c \cdot d$ is formed by a series of concatenations: $c_1 \cdot \tilde{c} \cdot c'_2 \cdot d_1 \cdot \tilde{d} \cdot d'_2$. Thus we have $c \cdot d = c_1 \cdot \tilde{c} \cdot \varphi_k(c'_2, d_1) \cdot \tilde{d} \cdot d'_2 + a$ with $a \in J_{k-2}$. This implies that

$$c \cdot d \equiv c_1 \otimes d'_2 \otimes \tilde{c} \varphi_k(c'_2, d_1) \tilde{d} \pmod{J_{k-2}}.$$

This proved the lemma. \square

By Lemma 4.1, we have an R -module decomposition: $\widehat{B_{n,R}} = V(n, n) \otimes V'(n, n) \otimes A_n \oplus V(n, n-2) \otimes V'(n, n-2) \otimes A_{n-2} \oplus V(n, n-4) \otimes V'(n, n-4) \otimes A_{n-4} \oplus \cdots \oplus V(n, \varepsilon) \otimes V'(n, \varepsilon) \otimes A_\varepsilon$, where ε is zero if n is even, and 1 if n is odd. The following lemma tells us how to get an ideal in $\widehat{B_{n,R}}$ from an ideal in the group algebras.

Lemma 4.3. *Let I be an ideal in A_k . Then $J_{k-2} + V(n, k) \otimes V'(n, k) \otimes I$ is an ideal in $\widehat{B_{n,R}}$.*

Proof. To prove the lemma, it is sufficient to show that for each $d = d_1 \otimes d'_2 \otimes \tilde{d}$ with $d \in \widehat{B_n}$ and $t(d) = l > k$, and $c = c_1 \otimes c'_2 \otimes \tilde{c}$ with $c \in \widehat{B_n}$ and $t(c) = k$, the following property holds:

$$(d_1 \otimes d'_2 \otimes \tilde{d})(c_1 \otimes c'_2 \otimes \tilde{c}) \equiv b \otimes c'_2 \otimes a \tilde{c} \pmod{J_{k-2}}$$

for some $b \in V(n, k)$; and the element a is an element in A_k which is independent of \tilde{c} . This property follows again by considering the composition of

tangles as in the proof of Lemma 4.2. In fact, the concatenation $d_1 \cdot \tilde{d} \cdot d'_2 \cdot c_1$ is an (n, m) -tangle with $m \leq k$. If $m < k$, then the composition of d and c is in J_{k-2} , so we are done. Suppose that $m = k$, then this concatenation $d_1 \cdot \tilde{d} \cdot d'_2 \cdot c_1$ is of the form $b \cdot a + a'$, where a' is a linear combination of (n, j) -tangles with $j < k$. If we concatenate this further with $\tilde{c} \cdot c'_2$, then we get the desired statement. \square

It is well-known how to define an involution i on $\hat{B}_{n,R}$: i is given by reflection of a diagram through its horizontal axis. Note also that there is an involution σ on A_n defined by $\sigma(w) = w^{-1}$ for any $w \in W$, where W is the extended affine Weyl group of type A_{n-1} . The following lemma easily follows from the fact that the extended affine Hecke algebra of type A is affine cellular as is shown in [KX3] and from [KX3, Lemma 2.4].

Lemma 4.4. *The group algebra A_n over \mathbb{Z} is an affine cellular \mathbb{Z} -algebra with respect to the involution σ .*

The following lemma describes the effect of i on an basis element of $\hat{B}_{n,R}$ and σ on the bilinear form φ_k , whose proof can be seen from the definition of i and the tangle concatenations.

Lemma 4.5. (1) *If $d = d_1 \otimes d'_2 \otimes \tilde{d}$, then $i(d) = d_2 \otimes d'_1 \otimes \sigma(\tilde{d})$.*

(2) *The involution σ on A_k and the bilinear form φ_k have the following property: $\sigma\varphi_k(c', d) = \varphi_k(d', c)$ for all $c, d \in D(n, k)$.*

Combined with Lemma 2.1, 4.2, 4.3, 4.4 and 4.5, we have proved the main result of this note.

Theorem 4.1. *The affine Brauer algebra $\hat{B}_{n,R}$ over a noetherian domain R is an affine cellular algebra with respect to the involution i .*

Let now A be an affine cellular algebra with a cell chain $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$, such that each subquotient J_i/J_{i-1} is an affine cell ideal of A/J_{i-1} . Then J_i/J_{i-1} is isomorphic to $\mathbb{A}(V_i, B_i, \varphi_i)$ for some free k -module V_i of finite rank, an affine k -algebra B_i and a k -bilinear form $\varphi_i : V_i \otimes_k V_i \rightarrow B_i$. Let (φ_{st}^i) be the matrix representing the bilinear form φ_i with respect to some choices of basis of V_i . Then Koenig and Xi obtain a parameterisation of simple modules over an affine cellular algebra by establishing a bijection between isomorphism classes of simple A -modules and the set

$$\{(j, m) | 1 \leq j \leq n, m \in \text{MaxSpec}(B_j) \text{ such that some } \varphi_{st}^i \notin m\},$$

where $\text{MaxSpec}(B_j)$ denotes the maximal ideal spectrum of B_j .

References

- [BCV] C. Bowman, A.G. Cox and M.De. Visscher, Decomposition numbers for the cyclotomic Brauer algebras in characteristic zero, *J. Algebra*. **378** (2013), 80-102.
- [BW] J.S. Birman and H. Wenzl, Braids, link polynomials and a new algebra, *Trans. Amer. Math. Soc.* **313** (1989), 249-273.
- [C1] W. Cui, Affine cellularity of BLN-algebras, preprint.
- [C2] W. Cui, Affine cellularity of affine q -Schur algebras, preprint.
- [E] J. Enyang, Cellular bases for the Brauer and Birman-Murakami-Wenzl algebras, *J. Algebra*. **281** (2004), 413-449.
- [G] F.M. Goodman, Cellularity of cyclotomic Birman-Wenzl-Murakami algebras, *J. Algebra*. **321** (2009), 3299-3320.
- [GH] F.M. Goodman and H. Hauschild, Affine Birman-Wenzl-Murakami algebras and tangles in the solid torus, *Fund. Math.* **190** (2006), 77-137.
- [GL] J.J. Graham and G.I. Lehrer, Cellular algebras, *Invent. Math.* **123** (1996), 1-34.
- [GM] F.M. Goodman and H.H. Mosley, Cyclotomic Birman-Wenzl-Murakami algebras I: Freeness and realization as tangle algebras, *J. Knot Theory Ramifications*. **18** (2009), 1089-1127.
- [GuM] J. Guilhot and V. Miemietz, Affine cellularity of affine Hecke algebras of rank two, *Math. Z.* **271** (2012), 373-397.
- [HO] R. Häring-Oldenburg, Cyclotomic Birman-Murakami-Wenzl algebras, *J. Pure Appl. Algebra*. **161** (2001), 113-144.
- [KL] A. Kleshchev and J. Loubert, Affine cellularity of Khovanov-Lauda-Rouquier algebras of finite types, <http://arXiv.org/1310.4467>.
- [KX1] S. Koenig and C. Xi, Cellular algebras: Inflation and Morita equivalences, *J. Lond. Math. Soc.* **60** (1999), 700-722.
- [KX2] S. Koenig and C. Xi, A characteristic free approach to Brauer algebras, *Trans. Amer. Math. Soc.* **353** (2001), 1489-1505.
- [KX3] S. Koenig and C. Xi, Affine cellular algebras, *Adv. Math.* **229** (2012), 139-182.
- [L] G. Lusztig, Aperiodicity in quantum affine \mathfrak{gl}_n , *Asian. J. Math.* **3** (1999), 147-177.
- [Mc] K. McGerty, Generalized q -Schur algebras and quantum Frobenius, *Adv. Math.* **214** (2007), 116-131.
- [Mu] J. Murakami, The Kauffman polynomial of links and representation theory, *Osaka J. Math.* **24** (1987), 745-758.

- [MW] H. Morton and A. Wassermann, A basis for the Birman-Wenzl algebra, unpublished manuscript, 1989, revised 2000, 1-29.
- [RX] H. Rui and J. Xu, On the semisimplicity of the cyclotomic Brauer algebras II, J. Algebra. **312** (2007), 995-1010.
- [RY] H. Rui and W. Yu, On the semisimplicity of the cyclotomic Brauer algebras, J. Algebra. **277** (2004), 187-221.
- [Xi] C. Xi, On the quasi-heredity of Birman-Wenzl algebras, Adv. Math. **154** (2000), 280-298.

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